$$v_e[t] = \text{idem}\{\mu \to \nu, y \to z, \rho^{(1)} \to \rho^{(2)}\}$$

b) if $\varepsilon^\circ = 0$, $t < \vartheta$, then

$$U_e^*(t, y, z, \lambda) = P(t, y), \quad V_e^*(t, y, z, \lambda) = Q(t, z)$$

The vector $l^\circ(\vartheta, t, y, z, \lambda)$, which furnishes a maximum to the right-hand side of Eq.(4.3), has the form

$$l^\circ = \frac{x}{\|x\|} + \lambda \frac{(\vartheta - t)^2}{\|x\|^3}[\|x\|^2(\nu z - \mu y) + x(-\nu x'z + \mu x'y)] + \ldots$$

The ES U_e and V_e determined in this manner furnish the approach game (4.1), (4.2) with a saddle point, and the game payoff $\varepsilon^\circ(\vartheta, t, y, z, \lambda)$ is given, for any position $\{t, y, z\}$, by the equation

$$\varepsilon^\circ = \|x\| - (\mu - \nu)(\vartheta - t) + \lambda[(\|z\|^2 - \|y\|^2)(\vartheta - t) + \|x\|^{-1}x'(\nu z - \mu y)(\vartheta - t)^2 + \frac{1}{3}(\nu^2 - \mu^2)(\vartheta - t)^3] + \ldots$$

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THE APPLICATION OF MONOMIAL LIE GROUPS TO THE PROBLEM OF ASYMPTOTICALLY INTEGRATING EQUATIONS OF MECHANICS*

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The basis of the algorithm of the asymptotic integration of equations of mechanics discussed below is the representation of the initial system as a monomial Lie group of transformations of the phase space into itself. Transformations of the system which reduce it to a simpler form are also sought in a class of systems possessing group properties. Matching the instrument of the analysis to the objective of the analysis enables us to limit the operations used in the algorithm to those from the corresponding operator algebra.

Hori's paper /1/, in which Lie series were used to construct an additional first integral in an autonomous Hamiltonian system, was followed by a number of papers which extended this approach to autonomous systems of general form (Hori, Kemel et al, a review of whose results can be found in /2, 3/). Note that all these papers are essentially only different forms of deriving Hausdorff's formula, which is well-known from the theory of Lie groups, complicated somewhat by the concept of parameter identification and order separation. Now results can only be obtained by refusing to consider systems of general form and by proceeding to

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analyse the more special types of systems that are characteristic for those or other areas of asymptotic theory, with the aim of improving existing procedures. One should proceed from Hausdorff's formula, without repeating its conclusion. That course is taken here, where we consider systems in so-called single-frequency standard form. This form of system is basic for the well-known Krylov-Hogolyubov method, which also achieves substantial simplification using group-theoretical principles.

The objective feature of that simplification are: 1) there is no need to solve the derivatives transformed at each step of the system, or to invert the equation of change; 2) the algorithm does not use power series of the small parameter, and all discussion of the procedure can be carried out in terms of the required asymptotic forms; 3) the expression for the arbitrary approximation can be obtained in the form of an explicit recurrence formula which is convenient when using computers that perform symbol calculations. Following /4/, the proposed algorithm can be generalized to multifrequency systems, essentially non-linear systems and resonance cases can be handled. Examples of the application of the method are considered.

Consider a system described by differential equations of the following form:

$$\frac{dx}{dt} = X(x, y, \varepsilon), \quad \frac{dy}{dt} = Y(x, y, \varepsilon) \quad (x \in R^1, y \in R^n, \varepsilon \ll 1) \tag{1}$$

where x is a scalar variable y is an n-dimensional vector and ε is a small parameter, the right-hand sides are analytic in some domain.

The presence of a small parameter enables us to use it effectively to form procedures for asymptotically constructing approximate solutions. At the same time the most productive approach is to construct solutions directly, and to reduce system (1) to a form that is more convenient for analysis (and also for solution) /5, 6/.

The methods of this reduction, which are fairly simple and convenient when constructing one or two approximations, become extremely cumbersome when the number of approximations is increased. The natural need arises to rationalize the respective procedures. The principle of this rationalization can be based on the idea of maximum agreement - with respect to groups of features - of the instrument of investigation and the objective of investigation. There are examples of this agreement in mechanics: linear systems are naturally transformed using linear substitutions; Euler's kinematic equations are non-linear and have singularities, but if we note that the rotations of a solid form the group $SO(3)$, the equations are obtained as linear in group variables, without singularities. The minimum-sized linear system is obtained in the notation of equations in quaternions, and if quaternions are also used when solving the equations, the operations performed do not exceed the limits of the operations of quaternion algebra and reduce to maximally simple and economic algorithms /7/.

System (1) with arbitrary non-linear right-hand sides nevertheless produces an extremely narrow class of mappings of the phase space into itself - the single-parameter Lie group /8, 9/ with the operator

$$A = X(x, y, \varepsilon)\, \partial/\partial x + Y(x, y, \varepsilon)\, \partial/\partial y \tag{2}$$

Therefore the transformation of system (1) should also be sought not in the class of arbitrary non-linear substitutions, as is usually done, but in the form of a single-parameter Lie group produced by some differential system which is defined in the same phase space as system (1)

$$dx/d\tau = M(x, y, \varepsilon), \; dy/d\tau = N(x, y, \varepsilon) \tag{3}$$

where τ is the parameter of the group, and its operator has the form

$$U = M(x, y, \varepsilon)\, \partial/\partial x + N(x, y, \varepsilon)\, \partial/\partial y \tag{4}$$

In this case the operations performed during the transformations will not exceed the limits of the Lie algebra operations of operators of the forms (2) and (4), which also promises substantial improvements, for unlike the linear objective (1) the objectives (2) and (4) are linear.

By a transformation of system (1) using the group specified by system (3) with the operator (4), we mean the change of variables $(x, y) \to (p, q)$: $p = f_1(x, y, \tau, \varepsilon)$, $q = f_2(x, y, \tau, \varepsilon)$, in which the functions f_1 and f_2 identically satisfy Eqs. (3), whilst $f_1(x, y, 0, \varepsilon) \equiv x$, $f_2(x, y, 0, \varepsilon) \equiv y$.

These changes of variables, and also the reverse changes, can be written in the form of Lie series using the operator (4):

$$p = x + \tau Ux + \frac{1}{2!}\tau^2 U^2 x + \ldots = e^{\tau U} x \tag{5}$$

$$q = y + \tau Uy + \frac{1}{2!}\tau^2 U^2 y + \ldots = e^{\tau U} y$$

$$x = p - \tau Up + \ldots = e^{-\tau U}p, \quad y = q - \tau Uq + \ldots = e^{-\tau U}q$$

At the same time in the case of a reverse change in the expression for the operator (Eq. (4)) instead of the variables x and y we should formally write the variables p and q:

$$U = M(p, q, \varepsilon)\,\partial/\partial p + N(p, q, \varepsilon)\,\partial/\partial q$$

The problem of the theory of perturbations, i.e. the aims of the transformations performed on system (1), can also be formulated in terms of groups. One of the basic facts of the group analysis of differential equations can be the basis of this formulation. If some single-parametric group of symmetries of system (1) is known, whose operator commutes with the operator of this system A, the latter can be reduced in order. When formulating problems of perturbation theory, assumptions that there are known solutions are made in relation to the unperturbed part of the system $(\varepsilon = 0)$. By analogy we shall also assume that the group of symmetries is only known for the degenerate $(\varepsilon = 0)$ system (1).

The formulation of the problem of perturbation theory takes the following form: group G of symmetries of system (1) when $\varepsilon = 0$ is given (G is the group operator), such that we have $[A, G]_{\varepsilon=0} = 0$. It is required to find the group U of transformations of system (1), that changes the operator A into the operator B, such that this group G is a group of symmetries of system (1) when $\varepsilon = \tau$: $[B, G]_{\tau=\varepsilon} = 0$. If this can be done, the order of system (1) can be reduced.

The brackets denote the operation of calculating the commutator which represents the linear operator calculated using the rule $[A, G] = AG - GA$.

Note that the traditional formulation of the problem in the method of averaging corresponds completely to the above. Indeed, in the method of averaging the degenerate system is autonomous and it is required to make the whole system autonomous. But this also means that the degenerate system permits a shift group with respect to t and it is required to transform the perturbed system such that it permits that group.

The above problem can be solved when the known group of symmetries is generated by the phase flow of the degenerate system: $G = A|_{\varepsilon=0}$, i.e. when the general solution of the system when $\varepsilon = 0$ is known. We shall construct an algorithm for the asymptotic solution of the problem of perturbations in this case. Without loss of generality we can assume that system (1) is written in canonical coordinates of group G and, consequently, has a well-known standard form:

$$dx/dt = 1 + \varepsilon X(x, y, \varepsilon), \quad dy/dt = \varepsilon Y(x, y, \varepsilon) \tag{6}$$

The standard form of the operator A also corresponds to it:

$$A = \partial/\partial x + \varepsilon\,[X(x, y, \varepsilon)\,\partial/\partial x + Y(x, y, \varepsilon)\,\partial/\partial y]$$

If A is the operator of the initial system, U is the operator of change of variables and B is the operator of transformations of the system, it is well-known that these three operators are connected by Cauchy's initial problem for Hausdorff's equation

$$dB/d\tau = [B, U]; \quad B_{\tau=0} = A$$

The solution of this problem, which gives an explicit connection between these operators, can be represented using the following series in powers of the parameter of the group τ:

$$B = A + \tau[A, U] + \frac{1}{2!}\tau^2[[A, U], U] + \ldots \tag{7}$$

If we subject the operator B to the commutation condition when $\tau = \varepsilon$ with G (we recall that $G = A_{\varepsilon=0} = \partial/\partial p$ and the commutation condition reduces to B being independent of p), we will obtain an equation for the operator U. We shall solve this equation asymptotically, for which we shall introduce the notation

$$A_k = A + o(\varepsilon^k), \quad B_k = B + o(\varepsilon^k), \quad U_k = U + o(\varepsilon^k)$$

where A_k, B_k, U_k are operators which differ from the exact operators by quantities of a higher order of smallness than ε^k.

The following chain of relations follows from Eq. (7) for the above operators:

$$B_0 = A_0, \quad B_1 = A_1 + \varepsilon[A_0, U_0], \ldots \tag{8}$$

$$B_n = A_n + \sum_{k=1}^{n} \frac{\varepsilon^k}{k!}\underbrace{[\ldots[}_{k}A_{n-k}, U_{n-k}], \ldots], U_{n-k}]$$

Consider the first commutator $(k = 1)$ in B_n:

$$[A_{n-1}, U_{n-1}] = [A_{n-1} - A_0, U_{n-1}] + [A_0, U_{n-1}] \tag{9}$$

Since $A_{n-1} - A_0 \sim \varepsilon$, then, without going outside the limits of the asymptotic form considered, instead of (9) we can write

$$[A_{n-1}, U_{n-1}] = [A_{n-1} - A_0, U_{n-2}] + [A_0, U_{n-1}] \Rightarrow B_n = \varepsilon[A_0, U_{n-1}] + L_n$$

where L_n is an operator that depends only on the asymptotic forms of the operator U which are the lowest in relation to U_{n-1}:

$$L_n = A_n + \varepsilon[A_{n-1} - A_0, U_{n-2}] + \sum_{k=2}^{n} \frac{\varepsilon^k}{k!} \underbrace{[\ldots[A_{n-k}, U_{n-k}], \ldots], U_{n-k}]}_{k} \tag{10}$$

Bearing in mind that $[A_0, U_k] = \partial U_k/\partial p$, we can rewrite Eq.(8) in the form

$$B_0 = A_0, \quad B_1 = \varepsilon \frac{\partial U_0}{\partial p} + A_1, \quad B_2 = \varepsilon \frac{\partial U_1}{\partial p} + A_2 + \tag{11}$$
$$\varepsilon[A_1 - A_0, U_0] + \frac{1}{2}\varepsilon^2[[A_0, U_0], U_0], \ldots,$$
$$B_n = \varepsilon \frac{\partial U_{n-1}}{\partial p} + L_n'$$

Eqs.(11) enable us to determine all the approximations for operator B and the operator of substitution U in succession.

Indeed, to construct the first approximation it is sufficient to take B_1 thus:

$$B_1 = \lim_{h\to\infty} \frac{1}{h} \int_0^h A_1\, dp \equiv \langle A_1 \rangle$$

Then U_0 is obtained by the quadrature:

$$U_0 = -\frac{1}{\varepsilon} \int (A_1 - \langle A_1 \rangle)\, dp \equiv -\frac{1}{\varepsilon} \int \bar{A}_1\, dp$$

The mean value with respect to p of the operator A_1 (provided that it exists) and the addition to the mean are denoted by $\langle A_1 \rangle$ and $\bar{A}$ respectively.

The choice of B_1 in the form of the mean from A_1 is dictated, on one hand, by the requirement that B must be independent of p, and on the other by the requirement that the equations of substitution are bounded with respect to p.

After the operator U_0 is obtained, we can proceed to construct the second approximation

$$B_2 = \left\langle A_2 + \varepsilon[A_1 - A_0, U_0] + \frac{1}{2}\varepsilon^2[[A_0, U_0], U_0] \right\rangle$$

which in turn enables us to obtain

$$U_1 = -\frac{1}{\varepsilon} \int \left(\bar{A}_2 + \overline{\varepsilon[A_1 - A_0, U_0]} + \frac{1}{2}\varepsilon^2 \overline{[[A_0, U_0], U_0]} \right) dp$$

Thus the general expression for an n-th approximation is obtained in the form

$$B_n = \langle L_n \rangle, \quad U_{n-1} = -\frac{1}{\varepsilon} \int \bar{L}_n\, dp \tag{12}$$

where the operator L_n is expressed in terms of the preceding approximations of the explicit finite formula (10). If the normal form converges, then an exact expression for the operator of the transformed system is $B = \lim \langle L_n \rangle$ as $n \to \infty$.

Since B_n, by construction, does not depend on p and $B_0 = A_0 = \partial/\partial p$, then

$$B_n = \frac{\partial}{\partial p} + \varepsilon \left[P_n(q, \varepsilon) \frac{\partial}{\partial p} + Q_n(q, \varepsilon) \frac{\partial}{\partial q} \right]$$

Consequently, Eqs.(6) acquire the following form in the variables (p, q)

$$dp/dt = 1 + \varepsilon P_n(q, \varepsilon), \quad dq/dt = \varepsilon Q_n(q, \varepsilon) \tag{13}$$

which also represents the final aim of the n-th approximation of the method of averaging.

It is necessary to substitute the solutions of system (13) into the equations of change

$$x = e^{-\varepsilon U_{n-2}} p, \quad y = e^{-\varepsilon U_{n-2}} q \tag{14}$$

in order to obtain a solution, in initial variables, of corresponding accuracy. A knowledge of the operator U_{n-1} is necessary to construct the $n+1$-th approximation, i.e. B_{n+1}.

Thus, the whole procedure reduces to calculating B_n and U_{n-1} using Eqs.(10) and (12). The expression for B_n determines the right-hand sides of system (13), and U_{n-2} determines the change of initial variables (x, y) into (p, q) in which the specified system (6) takes a form that does not depend on p, i.e. (13).

Since there is no averaging procedure in the above algorithm, the asymptotic estimates of the accuracy of this algorithm are equivalent to the usual accuracy estimates of the method of averaging /10/. However, in specific examples the real accuracy of the algorithm can be higher than that given by the method of averaging. This occurs because transformations which

are similar to (14) occur in the method of averaging, at the same time as the operators are constructed in the proposed algorithm. Therefore, if the exact expression of the operator can be obtained in a finite number of steps, Eqs.(14) also determine the expression for transformations which are, however, represented by infinite series. It is impossible to obtain them using a finite number of steps.

We shall illustrate the above with an example.

Example 1. Suppose it is required to use the change of variables $(x, y) \to (p, q)$ to reduce the following system to a form in which the dependence of the right-hand sides is eliminated:

$$dx/dt = 1, \quad dy/dt = -\varepsilon y^3 \cos^2 x$$

(it can be accurately integrated using the method of separation of variables). The transformation that reduces it to autonomous form can be written in the explicit form

$$x = p, \quad y = q\left(1 + \frac{1}{2}\varepsilon q^2 \sin 2p\right)^{-1/2} = \sum_{n=1}^{\infty} (-1)^n \varepsilon^n \frac{(2n-1)!!}{n!\,4^n} \sin^n 2p + q$$

However, it is impossible to obtain these formulas for a finite number of approximations using the Krylov-Bogolyubov method. Incidentally this accurate result is obtained in two approximations by means of the algorithm proposed above. The operator of this system, written using new variables, has the form

$$A = \partial/\partial p - \varepsilon q^3 \cos^2 p\, \partial/\partial q$$

The application of the above procedure gives

$$B_1 = \langle A_1 \rangle = \frac{\partial}{\partial p} - \frac{1}{2}\varepsilon q^3 \frac{\partial}{\partial q}$$

$$U_0 = \left(\frac{q^3}{2}\int \cos 2p\, dp\right)\frac{\partial}{\partial q} = \frac{1}{4} q^3 \sin 2p \frac{\partial}{\partial q}$$

The second approximation

$$B_2 = \langle A_2 + \varepsilon [A_1 - A_0, U_0] + {}^1/_2\varepsilon^2 [[A_0, U_0], -U_0] \rangle$$

We shall calculate the commutators

$$[A_1 - A_0, U_0] = \left[-\varepsilon q^3 \cos^2 p \frac{\partial}{\partial q}, \quad \frac{1}{4} q^3 \sin 2p \frac{\partial}{\partial q}\right] = 0$$

$$[[A_0, U_0], U_0] = \left[\frac{\partial U_0}{\partial p}, U_0\right] = \left[\frac{1}{2} q^3 \cos 2p \frac{\partial}{\partial q}, \frac{1}{4} \sin 2p \frac{\partial}{\partial q}\right] = 0$$

Consequently

$$B_2 = \langle A_2 \rangle = \frac{\partial}{\partial p} - \varepsilon \frac{q^3}{2}\frac{\partial}{\partial q} = B_1, \quad U_1 = U_0 = \frac{1}{4} q^3 \sin 2p \frac{\partial}{\partial q}$$

Thus, the second approximation agrees with the first. It is obvious from Eq.(10) that the same will also be obtained for any approximation. The problem is thereby solved exactly.

The transformed equations have the form

$$dp/dt = 1, \quad dq/dt = -\frac{1}{2}\varepsilon q^3$$

The connection between the new variables and the old ones is obtained using (14) and agrees with the above exact solution.

Example 2 /11/. Consider the equation $\varphi'' + \varphi - a\varphi^2 = \mu \cos t$. In this equation there is a principal resonance, and the periodic solution decomposes into a series in fractional powers of the small parameter. It was stated in this connection /11/ that these solutions cannot be obtained using the theory of quasilinear system. We shall show that this is not so. The occurrence of fractional powers is only connected with the choice of the scale of measurement of the variables and is in no way determined by the asymptotic procedure for constructing a solution.

In order to use the quasilinear approach, we shall introduce a small scale in the written equation using the formula $\varphi = \varepsilon z$ (ε is the small parameter) and rewrite it in the form of a quasilinear equation

$$z'' + z = \varepsilon a z^2 + \mu\varepsilon^{-1} \cos t$$

For the non-linear and inhomogeneous terms to have the same order of influence on the oscillator, we must assume $\mu\varepsilon^{-1} = \varepsilon^2$.

We shall proceed in this equation to the Van der Pol variables (canonical coordinates in the phase space (t, z, z') of the group of screws generated by the phase flow of a degenerate system)

$$t = x, \quad z = y_1 \sin x + y_2 \cos x, \quad z' = y_1 \cos x - y_2 \sin x$$

The initial equation under the additional equation $y_1' \sin x + y_2' \cos x = 0$ is rewritten in

the form of a system of standard form:

$$x^{\cdot} = 1, \quad y_1^{\cdot} = \zeta \cos x, \quad y_2^{\cdot} = -\zeta \sin x$$
$$\zeta = \varepsilon a (y_1 \sin x + y_2 \cos x)^2 + \varepsilon^2 \cos x$$

The operator of this system in the new variables:

$$A = A_2 = \frac{\partial}{\partial p} + \{\varepsilon a (q_1 \sin p + q_2 \cos p)^2 + \varepsilon^2 \cos p\} \left(\cos p \frac{\partial}{\partial q_1} - \sin p \frac{\partial}{\partial q_2}\right)$$
$$A_0 = \frac{\partial}{\partial p}, \quad A_1 = \frac{\partial}{\partial p} + \varepsilon a (q_1 \sin p + q_2 \cos p)^2 \left(\cos p \frac{\partial}{\partial q_1} - \sin p \frac{\partial}{\partial q_2}\right)$$

The first approximation

$$B_1 = \langle A_1 \rangle = A_0 = \frac{\partial}{\partial p}, \quad U_0 = -\frac{1}{\varepsilon} \int \bar{A}_1 \, dp =$$
$$a \left\{\frac{1}{3}(q_2^2 - q_1^2) \sin^3 p + \frac{2}{3} q_1 q_2 \cos^3 p - q_2^2 \sin p\right\} \frac{\partial}{\partial q_1} +$$
$$a \left\{\frac{1}{3}(q_1^2 - q_2^2) \cos^3 p + \frac{2}{3} q_1 q_2 \sin^3 p - q_1^2 \cos p\right\} \frac{\partial}{\partial q_2}$$

The second approximation. Since $A_1 - A_0 = -\varepsilon [A_0, U_0]$, we obtain

$$B_2 = \langle A \rangle + \frac{\varepsilon}{2} \langle [A_1 - A_0, U_0] \rangle =$$
$$\frac{\partial}{\partial p} + \frac{1}{2} \varepsilon^2 \left\{1 + \frac{5}{6} a^2 q_2 (q_1^2 + q_2^2)\right\} \frac{\partial}{\partial q_1} - \frac{5}{12} \varepsilon^2 a^2 q_1 (q_1^2 + q_2^2) \frac{\partial}{\partial q_2}$$

The system of the second approximation with separate variables is obtained in the form

$$\frac{dp}{dt} = 1, \quad \frac{dq_1}{dt} = \frac{\varepsilon^2}{2} \left\{1 + \frac{5}{6} a^2 q_2 (q_1^2 + q_2^2)\right\}, \quad \frac{dq_2}{dt} = -\frac{5}{12} \varepsilon^2 a^2 q_1 (q_1^2 + q_2^2) \tag{15}$$

The connection between the initial variables and the new ones is

$$x = p, \quad y_1 = q_1 - \varepsilon U_0 q_1, \quad y_2 = q_2 - \varepsilon U_0 q_2 \tag{16}$$

Solving system (15) and substituting the result into (16), we obtain a solution of the problem in initial variables. If, as in /11/), only the periodic solution is of interest, then, determining the stationary point $q_1 = 0, q_2 = -(6/(5a^2))^{1/2}$ from (15), we obtain an expression that agrees with that in /11/ for the variable φ.

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